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ORIGINAL RESEARCH

Fuzzy Confidence Intervals by the Likelihood Ratio: Testing Equality of Means—Application on Swiss SILC Data

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Abstract

We propose a practical procedure of construction of fuzzy confidence intervals by the likelihood method where the observations and the hypotheses are considered to be fuzzy. We use the bootstrap technique to estimate the distribution of the likelihood ratio. The chosen bootstrap algorithm consists on randomly drawing observations by preserving the location and dispersion measures of the original fuzzy data set. A metric $d^{\theta^\star}_{SGD}$ based on the well-known signed distance measure is considered in this case. We expose a simulation study to investigate the influence of the fuzziness of the computed maximum likelihood estimator on the constructed confidence intervals. Based on these intervals, we introduce a hypothesis test for the equality of means of two groups with its corresponding decision rule. The highlight of this paper is the application of the defended approach on the Swiss SILC Surveys. We empirically investigate the influence of the fuzziness vs. the randomness of the data as well as of the maximum likelihood estimator on the confidence intervals. In addition, we perform an empirical analysis where we compare the mean of the group "Swiss nationality" to the group "Other nationalities" for the variables Satisfaction of health situation and Satisfaction of financial situation.

Keywords Bootstrap technique · Likelihood ratio · Fuzzy confidence interval · Fuzzy statistics · Fuzzy hypotheses · Equality of means · Fuzzy data · Statistical inference · Fuzzy analysis of variance (FANOVA)

Introduction and Motivation

A typical hypothesis testing procedure can be accomplished by, for example, constructing confidence intervals for a particular parameter. This method is widely used in practice. However, once we consider the data and/or the hypotheses to be fuzzy, the corresponding statistical methods have to be updated. Some approaches already exist in the theory of fuzzy sets. For instance, Kruse and Meyer [17] presented a theoretical definition of fuzzy confidence intervals. Several researchers have afterwards proposed refined definitions of fuzzy confidence intervals. For instance, Viertl and Yeganeh [22] proposed a definition of the so-called confidence regions. Their main application was in the Bayesian context. Kahraman et al. [16] described some approaches to the construction of fuzzy confidence intervals, as well as the concept of hesitant fuzzy confidence intervals. Couso and Sanchez [9] provided an approach that considers the inner and outer approximations of confidence intervals in the context of fuzzy observations. Unfortunately, these various approaches are limited because they were all conceived to test a specific parameter with a pre-defined distribution. It would therefore be advantageous to develop a unified general approach to fuzzy confidence intervals.

In classical statistics, the likelihood ratio method is considered an alternative tool for the construction of confidence intervals. In the fuzzy environment, this method using uncertain data has multiple advantages.

Gil and Casals [13] used the likelihood ratio in a hypothesis testing procedure where fuzziness is contained in the data. In Berkachy and Donzé [5], we proposed a practical procedure to construct confidence intervals by the likelihood ratio method which is seen in some sense general. The

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procedure can be easily adapted to specific cases. However, the distribution of the likelihood ratio is a priori unknown and has to be estimated or derived from strong assumptions. Under classical assumptions, we note that this ratio is known to be χ^2-distributed with degrees of freedom corresponding to the number of constraints applied to parameters. We propose to use the bootstrap technique extended to the fuzzy environment to estimate the distribution of the likelihood ratio. A main contribution of Berkachy and Donzé [7] is to provide two algorithms to constitute the bootstrapped samples mainly using the location and dispersion characteristics calculated based on a new version of the signed distance measure written as the $d^{\theta\star}_{SGD}$ metric and detailed in Berkachy [1]. We highlight that the Expectation-Maximization (EM) algorithm based on the fuzziness of data described by Denoeux [10] is used to calculate the maximum likelihood estimators (ML-estimators).

The defended procedure is considered efficient and computationally light because we do not have to consider every single value of the support set of the involved fuzzy numbers, as in the traditional fuzzy method. Indeed, four conveniently chosen values are used in the construction process. The presented calculations are done using the `R` package `FuzzySTs` shown in [8]. We propose to use our fuzzy confidence interval to test the equality of means. We expose a procedure and give the corresponding decision rule. An application on Swiss SILC data, described in [21], give us the opportunity to apply and test our methods. Consideration on sensitivity and robustness are also shown.

The paper is organised as follows. We open the paper in "Definitions" with fundamental definitions of fuzziness. In "The signed distance", we present the definition of the signed distance measure, followed by the definition of the $d^{\theta\star}_{SGD}$ metric in "The $d^{\theta\star}_{SGD}$ Metric". "Traditional fuzzy confidence intervals" is devoted to the construction of the traditional fuzzy confidence intervals. In "Fuzzy confidence intervals by the likelihood method", we discuss our concept of fuzzy confidence intervals constructed using the likelihood method and detail the bootstrap algorithms to approximate the distribution of the likelihood ratio. In addition, a simulation study illustrates the proposed algorithms. We end the paper with "Application on SILC 2017" by the application on the Swiss SILC data.

Definitions

Let us first expose the basic definitions and concepts of fuzziness.

Definition 1 (Fuzzy set) If A is a collection of objects denoted generically by x, then a fuzzy set or class $\widetilde{X}$ in A is a set of ordered pairs:

$$\widetilde{X} = \left\{(x, \mu_{\widetilde{X}}(x)) \mid x \in A\right\}, \tag{1}$$

where the mapping $\mu_{\widetilde{X}}$ representing the "grade of membership" is a crisp real valued function such that

$$\begin{aligned}\mu_{\widetilde{X}} : \mathbb{R} &\to [0;1]\\ x &\mapsto \mu_{\widetilde{X}}(x)\end{aligned}$$

is called the membership function.

It is useful to show the support and the kernel of a given fuzzy set. They are given as follows:

Definition 2 (Support and kernel of a fuzzy set) The support and the kernel of a fuzzy set $\widetilde{X}$ denoted respectively by supp $\widetilde{X}$ and core $\widetilde{X}$, are given by:

$$\text{supp}\ \widetilde{X} = \left\{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) > 0\right\}, \tag{2}$$

$$\text{core}\ \widetilde{X} = \left\{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) = 1\right\}. \tag{3}$$

In other terms, the support of a fuzzy set $\widetilde{X}$ is a crisp set containing all the elements such that their membership function is not zero. In the same manner, the core of the fuzzy set $\widetilde{X}$ is a crisp set containing all elements with degree of membership equal to one.

We often characterize a given fuzzy set by a collection of crisp sets called the α-level sets. They are given in the following manner:

Definition 3 (α-level set or α-cut) An α-level set $\widetilde{X}_\alpha$ of the fuzzy set $\widetilde{X}$ is the (crisp) set of elements such that:

$$\widetilde{X}_\alpha = \left\{x \in A \mid \mu_{\widetilde{X}}(x) \geq \alpha\right\}. \tag{4}$$

The α-level set is a closed bounded and non-empty interval denoted generally by $\left[\widetilde{X}^L_\alpha; \widetilde{X}^R_\alpha\right]$ where for $\forall\alpha \in [0;1]$, $\widetilde{X}^L_\alpha$ and $\widetilde{X}^R_\alpha$ are the left and right hand sides of $\widetilde{X}_\alpha$ called respectively the left and right α-cuts such that:

$$\widetilde{X}^L_\alpha = \inf\left\{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) \geq \alpha)\right\} \text{ and } \widetilde{X}^R_\alpha = \sup\left\{x \in \mathbb{R} \mid \mu_{\widetilde{X}}(x) \geq \alpha)\right\}. \tag{5}$$

Furthermore, a fuzzy number $\widetilde{X}$, also called Left–Right (L–R) fuzzy number, can be represented by the family set of his α-cuts $\left\{\widetilde{X}_\alpha \mid \alpha \in [0;1]\right\}$. This set is a union of finite compact and bounded intervals $\left[\widetilde{X}^L_\alpha(\alpha); \widetilde{X}^R_\alpha(\alpha)\right]$ such that, $\forall\alpha \in [0;1]$,

$$\widetilde{X} = \bigcup_{0\leq\alpha\leq 1}\left[\widetilde{X}^L_\alpha(\alpha); \widetilde{X}^R_\alpha(\alpha)\right], \tag{6}$$

where $\widetilde{X}^L_\alpha(\alpha)$ and $\widetilde{X}^R_\alpha(\alpha)$ are the functions of the left and right hand sides of $\widetilde{X}$.

Remark 1 For sake of simplicity, common shapes of L–R fuzzy numbers are often used in practice. We particularly mention triangular fuzzy numbers denoted by a triplet as $\widetilde{X} = (p, q, r)$, with p, q and $r \in \mathbb{R}$, and trapezoidal fuzzy numbers denote by a quadruple as $\widetilde{X} = (p, q, r, s)$, with p, q, r, and $s \in \mathbb{R}$.

The Signed Distance

The signed distance measure was firstly used in the context of ranking fuzzy numbers by Yao and Wu [23]. It has also served in some other contexts: Berkachy and Donzé [3] used it in the assessment of linguistic questionnaires; Berkachy and Donzé [6] used it in hypotheses testing; etc. Although this measure is considered to be simple in terms of computations, it has interested specialists because of its directionality. This latter means that it can be positive or negative, indicating the direction between two particular fuzzy numbers. From another side, Dubois and Prade [11] described it as the expected value of a given fuzzy number. This measure is briefly written as follows:

Definition 4 (Signed distance between two fuzzy sets)
Let $\widetilde{X}$ and $\widetilde{Y}$ be two sets of the class of fuzzy sets $\mathbb{F}(\mathbb{R})$. Their respective α-cuts are written as $\widetilde{X}_\alpha$ and $\widetilde{Y}_\alpha$ such that their left and right α-cuts denoted respectively by $\widetilde{X}_\alpha^L$, $\widetilde{X}_\alpha^R$, $\widetilde{Y}_\alpha^L$ and $\widetilde{Y}_\alpha^R$ are integrable for all $\alpha \in [0;1]$.

The signed distance d_{SGD} between $\widetilde{X}$ and $\widetilde{Y}$ is the mapping

$$d_{SGD} : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to \mathbb{R}$$
$$\widetilde{X} \times \widetilde{Y} \mapsto d_{SGD}(\widetilde{X}, \widetilde{Y}),$$

such that

$$d_{SGD}(\widetilde{X}, \widetilde{Y}) = \frac{1}{2} \int_0^1 \left[\widetilde{X}_\alpha^L(\alpha) + \widetilde{X}_\alpha^R(\alpha) - \widetilde{Y}_\alpha^L(\alpha) - \widetilde{Y}_\alpha^R(\alpha) \right] \mathrm{d}\alpha. \tag{7}$$

We are often interested by the signed distance of a particular fuzzy number measured from the fuzzy origin $\widetilde{0}$ as follows:

Definition 5 (Signed distance of a fuzzy set)
The signed distance of the fuzzy set $\widetilde{X}$ measured from the fuzzy origin $\widetilde{0}$ is given by:

$$d_{SGD}(\widetilde{X}, \widetilde{0}) = \frac{1}{2} \int_0^1 \left[\widetilde{X}_\alpha^L(\alpha) + \widetilde{X}_\alpha^R(\alpha) \right] \mathrm{d}\alpha. \tag{8}$$

The $d_{SGD}^{\theta^\star}$ Metric

Although the signed distance d_{SGD} is seen as advantageous in terms of simplicity and accessibility, it presents also important drawbacks as detailed in Berkachy [2]. The major ones are given as follows:

1. Mainly because of its directionality, this distance cannot be defined as a full metric. It lacks topological characteristics, such as separability and symmetry.
2. It coincides with a central location measure. Thus, this distance depends strongly on its extreme values. In other words, neither the inner points between the extreme values nor the shape of the fuzzy numbers could affect this measure.

For these reasons, we propose a new L_2 metric denoted by $d_{SGD}^{\theta^\star}$. It is seen as a generalisation of the signed distance d_{SGD}. This new metric depends on a weight parameter called $\theta^\star$. Using $d_{SGD}^{\theta^\star}$, we take into account the deviation in the shapes and its possible irregularities from one side, and the central location measure from another one. This measure has the necessary and sufficient conditions to constitute a metric of fuzzy quantities as proved in Berkachy [2]. Let us first define the so-called deviations of the shape of a given fuzzy number written in terms of the distance d_{SGD}:

Definition 6 (Left and right deviations [2])
Consider $\widetilde{X}$ to be a fuzzy number with its α-level set $\widetilde{X}_\alpha = [\widetilde{X}_\alpha^L, \widetilde{X}_\alpha^R]$, $\widetilde{X} \in \mathbb{F}(\mathbb{R})$. The left and right deviations of the shape of $\widetilde{X}$ denoted by $\mathrm{dev}^L\widetilde{X}$ and $\mathrm{dev}^R\widetilde{X}$ can be written by:

$$\mathrm{dev}^L\widetilde{X}(\alpha) = d_{SGD}(\widetilde{X}, \widetilde{0}) - \widetilde{X}_\alpha^L, \tag{9}$$

$$\mathrm{dev}^R\widetilde{X}(\alpha) = \widetilde{X}_\alpha^R - d_{SGD}(\widetilde{X}, \widetilde{0}), \tag{10}$$

where $d_{SGD}(\widetilde{X}, \widetilde{0})$ is the signed distance of $\widetilde{X}$ measured from the fuzzy origin $\widetilde{0}$.

We define now the new metric $d_{SGD}^{\theta^\star}$ as follows:

Definition 7 (The $d_{SGD}^{\theta^\star}$ distance [2])
Consider two fuzzy numbers $\widetilde{X}$ and $\widetilde{Y}$ of the class of non-empty compact and bounded fuzzy numbers. Let $\theta^\star$ be the weight chosen for the modelling of the shape of these fuzzy numbers such that $0 \le \theta^\star \le 1$. Based on the signed distance between $\widetilde{X}$ and $\widetilde{Y}$, the L_2 metric $d_{SGD}^{\theta^\star}$ is the mapping

$$d_{SGD}^{\theta^\star} : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to \mathbb{R}^+$$
$$\widetilde{X} \times \widetilde{Y} \mapsto d_{SGD}^{\theta^\star}(\widetilde{X}, \widetilde{Y}),$$

such that

$$d_{SGD}^{\theta^\star}(\widetilde{X}, \widetilde{Y}) = \Big(\big(d_{SGD}(\widetilde{X}, \widetilde{Y})\big)^2 + \theta^\star \Big(\int_0^1 \max\big(\mathrm{dev}^R \widetilde{Y}(\alpha) - \mathrm{dev}^L \widetilde{X}(\alpha), \mathrm{dev}^R \widetilde{X}(\alpha) - \mathrm{dev}^L \widetilde{Y}(\alpha)\big) \mathrm{d}\alpha \Big)^2 \Big)^{\frac{1}{2}}. \tag{11}$$

It is important at this stage to show a direct mathematical relationship between the $d_{SGD}^{\theta^\star}$ metric and the signed distance d_{SGD}. Therefore, let us recall the concept of the nearest trapezoidal symmetrical fuzzy number. Further information in addition to the detailed proof, are shown in [2]. Remark that this concept will be used in the process of generation of random samples in the forthcoming sections.

Definition 8 (Nearest trapezoidal fuzzy number [2])

The nearest symmetrical trapezoidal fuzzy number $\widetilde{S}$ written by the quadruple $\widetilde{S} = [s_0 - 2\epsilon, s_0 - \epsilon, s_0 + \epsilon, s_0 + 2\epsilon]$ to a fuzzy number $\widetilde{X}$ with respect to the metric $d_{SGD}^{\theta^\star}$ is given such that

$$s_0 = d_{SGD}\big(\widetilde{X}, \widetilde{0}\big), \tag{12}$$

$$\epsilon = \frac{9}{14} d_{SGD}\big(\widetilde{X}, \widetilde{0}\big) - \frac{3}{7} \int_0^1 \widetilde{X}_\alpha^L (2 - \alpha) \mathrm{d}\alpha. \tag{13}$$

Fuzzy Confidence Intervals for a Pre-Defined Parameter

A fuzzy confidence interval is a very great tool for statistical inference. We first present the definition of a traditional fuzzy confidence intervals in "Traditional fuzzy confidence intervals", followed by our procedure of estimation of intervals by the likelihood ratio method and using the bootstrap technique in "Fuzzy confidence intervals by the likelihood method". Based on the designed confidence intervals, we also introduce a hypotheses test for the equality of means. A simulation study is finally provided in "Simulation study".

Traditional Fuzzy Confidence Intervals

A given confidence interval is often produced for a particular parameter denoted by θ. In an epistemic approach, this interval is considered to be fuzzy. This fuzziness is a direct consequence of the fuzziness of the considered parameter. Kruse and Meyer [17] proposed a main approach to write a fuzzy confidence intervals in such conditions. Many procedures have been derived to compute this interval. A known one relies on considering a pre-defined distribution as seen in the following construction procedure:

First, let $X_1, \ldots, X_n$ be a random sample of size n. We consider this sample to be fuzzy, and we call $\widetilde{X}_1, \ldots, \widetilde{X}_n$ its fuzzy perception. For a particular parameter denoted by θ, we are interested in testing the following hypotheses:

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0.$$

To accomplish this task, an idea could be to construct a fuzzy confidence interval for θ at a given significance level δ. Based on [17], a two-sided fuzzy confidence interval $\widetilde{\Pi}$ for θ is defined by:

Definition 9 (Fuzzy confidence interval [17])

Let $[\pi_1, \pi_2]$ be a symmetrical confidence interval for a particular parameter θ at the significance level δ. A fuzzy confidence interval $\widetilde{\Pi}$ is a convex and normal fuzzy set such that its left and right α-cuts, respectively written by $\widetilde{\Pi}_\alpha = [\widetilde{\Pi}_\alpha^L, \widetilde{\Pi}_\alpha^R]$, are written in the following manner:

$$\widetilde{\Pi}_\alpha^L = \inf\big\{a \in \mathbb{R} : \exists x_i \in (\widetilde{X}_i)_\alpha, \forall i = 1, \ldots, n, \text{ such that } \pi_1(x_1, \ldots, x_n) \leq a\big\},$$
$$\widetilde{\Pi}_\alpha^R = \sup\big\{a \in \mathbb{R} : \exists x_i \in (\widetilde{X}_i)_\alpha, \forall i = 1, \ldots, n, \text{ such that } \pi_2(x_1, \ldots, x_n) \geq a\big\}.$$

The constructed fuzzy confidence interval has a confidence of $1 - \delta$ if for a parameter θ, the equation

$$P\big(\widetilde{\Pi}_\alpha^L \leq \theta \leq \widetilde{\Pi}_\alpha^R\big) \geq 1 - \delta, \quad \forall \alpha \in [0;1] \tag{14}$$

is verified. A one-sided fuzzy confidence interval is likewise conceivable. A left one-sided fuzzy confidence interval at a confidence level $1 - \delta$ denoted by $\widetilde{\Pi}_\alpha$ is written by its α-level sets as follows:

$$\widetilde{\Pi}_\alpha = [\widetilde{\Pi}_\alpha^L, \infty].$$

In the same way, the α-cuts of a right one-sided one are written by:

$\widetilde{\Pi}_\alpha = [-\infty, \widetilde{\Pi}_\alpha^R]$.

Detailed examples illustrating this definition can be found in Berkachy [2] and Berkachy and Donzé [7].

Fuzzy Confidence Intervals by the Likelihood Method

We presented in Berkachy and Donzé [7] and Berkachy and Donzé [5] a generalisation of the traditional construction procedure. The aim was to show a practical tool based on the concept of likelihood ratio method to estimate fuzzy confidence intervals, in which the fuzziness contained in the variables is conveniently taken into consideration. We highlight that the likelihood ratio is a common tool in classical statistics as well. In the fuzzy environment, Gil and Casals [13] as instance used it in the context of hypotheses testing. In this section, we briefly recall the defended procedure. Note that further detailed information can be found in Berkachy and Donzé [7] and Berkachy [2].

We first define the likelihood function of a fuzzy observation. Consider $\widetilde{X}_i$ to be a fuzzy variable, with its fuzzy perception. Therefore, we denote by $\widetilde{X}_i$ a fuzzy random variable (FRV) such that its corresponding fuzzy realisation $\widetilde{x}_i$ is associated with a measurable membership function given by $\mu_{\widetilde{x}_i}$ in the sense of Borel, i.e. $\mu_{\widetilde{x}_i} : x \to [0;1]$. Based on the probability concepts proposed in Zadeh [24], the likelihood function described in the fuzzy context can be expressed by:

Definition 10 (Likelihood function of a fuzzy observation) Let $\underline{\widetilde{\theta}}$ be a vector of fuzzy parameters in the parameter space Θ. For a single fuzzy observation $\widetilde{x}_i$, the likelihood function can be given by:

$$L(\underline{\widetilde{\theta}};\widetilde{x}_i) = P(\widetilde{x}_i;\underline{\widetilde{\theta}}) = \int_{\mathbb{R}} \mu_{\widetilde{x}_i}(x) f(x;\underline{\widetilde{\theta}})dx. \tag{15}$$

This probability can also be written using the α-cuts of the involved fuzzy numbers.

Let now $\widetilde{\underline{x}}$ be a fuzzy sample composed of all the fuzzy realisations $\widetilde{x}_i$ of the fuzzy random variables $\widetilde{X}_i, i = 1, \ldots, n$. The corresponding likelihood function $L(\underline{\widetilde{\theta}};\underline{\widetilde{x}})$ can then be given by:

$$L(\underline{\widetilde{\theta}};\underline{\widetilde{x}}) = P(\underline{\widetilde{x}};\underline{\widetilde{\theta}}) \tag{16}$$

$$= \int_{\mathbb{R}} \mu_{\widetilde{x}_1}(x) f(x;\underline{\widetilde{\theta}})dx \cdot \ldots \cdot \int_{\mathbb{R}} \mu_{\widetilde{x}_n}(x) f(x;\underline{\widetilde{\theta}})dx \tag{17}$$

$$= \prod_{i=1}^{n} \int_{\mathbb{R}} \mu_{\widetilde{x}_i}(x) f(x;\underline{\widetilde{\theta}})dx. \tag{18}$$

It is then important to write the log-likelihood function $l(\underline{\widetilde{\theta}};\underline{\widetilde{x}})$ as follows:

$$\begin{aligned} l(\underline{\widetilde{\theta}};\underline{\widetilde{x}}) &= \log L(\underline{\widetilde{\theta}};\underline{\widetilde{x}}) \\ &= \log \int_{\mathbb{R}} \mu_{\widetilde{x}_1}(x) f(x;\underline{\widetilde{\theta}})dx + \cdots + \log \int_{\mathbb{R}} \mu_{\widetilde{x}_n}(x) f(x;\underline{\widetilde{\theta}})dx. \end{aligned} \tag{19}$$

Now consider $\underline{\widehat{\widetilde{\theta}}}$ the maximum likelihood estimator (ML-estimator) of the fuzzy parameter $\underline{\widetilde{\theta}}$. The likelihood ratio is written by:

$$\frac{L(\underline{\widetilde{\theta}};\underline{\widetilde{x}})}{L(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}})},$$

such that $L(\underline{\widetilde{\theta}};\underline{\widetilde{x}})$ is the likelihood function related to the fuzzy parameter $\underline{\widetilde{\theta}}$, and $L(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}})$ is the likelihood function evaluated at the estimator $\underline{\widehat{\widetilde{\theta}}}$. It is essential at this stage to write the logarithm of this ratio, given also by the difference between the log-likelihood functions evaluated at $\underline{\widehat{\widetilde{\theta}}}$ and at $\underline{\widetilde{\theta}}$. Therefore, the statistic LR can be given in the following manner:

$$LR = -2 \log \frac{L(\underline{\widetilde{\theta}};\underline{\widetilde{x}})}{L(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}})} = 2\left[l(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}}) - l(\underline{\widetilde{\theta}};\underline{\widetilde{x}})\right], \tag{20}$$

such that $L(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}}) \neq 0$, $L(\underline{\widetilde{\theta}};\underline{\widetilde{x}}) \neq 0$ and are both finite.

Under classical statistical assumptions in the crisp case, the ratio LR is proven to be asymptotically χ^2-distributed with a given number of degrees of freedom. In the fuzzy statistical theory, a main issue is that we do not have any proven asymptotic property for the distribution of this ratio. Hence, we propose to solve this problem using the so-called bootstrap techniques.

We remind that constructing a $100(1-\delta)\%$ confidence interval means to find every value of $\widetilde{\theta}$ for which we reject or we do not reject the null hypothesis H_0. For this construction, consider η to be the $(1-\delta)$-quantile of the distribution of the statistic LR. We could then write the confidence interval by:

$$2\left[l(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}}) - l(\underline{\widetilde{\theta}};\underline{\widetilde{x}})\right] \leq \eta. \tag{21}$$

This latter is equivalent to

$$l(\underline{\widetilde{\theta}};\underline{\widetilde{x}}) \geq l(\underline{\widehat{\widetilde{\theta}}};\underline{\widetilde{x}}) - \frac{\eta}{2}. \tag{22}$$

In other terms, the constructed interval has to be composed of all possible values of $\widetilde{\theta}$, for which the log-likelihood maximum varies by $\frac{\eta}{2}$ at most. Based on the statistic LR, a mandatory condition for this construction is that for every value of the parameter θ, the fuzzy confidence interval by the likelihood ratio $\widetilde{\Pi}_{LR}$ given by its left and right α-cuts $[(\widetilde{\Pi}_{LR})_\alpha^L; (\widetilde{\Pi}_{LR})_\alpha^R]$ has to verify the following equation

$$P\left((\widetilde{\Pi}_{LR})_{\alpha}^{L} \leq \theta \leq (\widetilde{\Pi}_{LR})_{\alpha}^{R}\right) \geq 1-\delta, \forall \alpha \in [0;1]. \tag{23}$$

To insure the above-mentioned conditions, we propose the following procedure of construction of fuzzy confidence intervals as seen in [7] and [2].

Procedure

We propose a revisited approach of construction of fuzzy confidence intervals by the likelihood ratio method, in which the data set is assumed to be vague. Consequently, the log-likelihood function becomes fuzzy-dependent, as well as the considered parameter. We could directly deduce that the needed ML-estimator have to be also fuzzy. Assume then that the calculated crisp ML-estimator is modelled by a well-chosen fuzzy number. It is natural to see that the support set of this fuzzy number is the set of crisp elements. Therefore, every element of this set have to be accordingly used in the calculation process of the log-likelihood function. However, this task seems to be very tedious because of the computational burden of such process. Thereby, we suggest to choose specific values which triggers the calculation of the so-called threshold points. The process of calculating the fuzzy confidence interval will be based on the intersection between these threshold points and the log-likelihood curve. The complete description of the procedure is as follows:

First, let us expose the so-called standardising function. This latter is intentionally proposed to preserve the [0; 1]-interval identity as a basic property of α-level sets. It is written as:

Definition 11 (Standardising function [1])

Consider a value θ contained in the the support set of a fuzzy number $\widetilde{\theta}$, i.e. $\theta \in$ supp $(\widetilde{\theta})$. The standardising function I_{stand} is given by:

$$I_{\text{stand}} : \quad \mathbb{R} \to \mathbb{R}$$
$$l(\theta, \underline{\widetilde{x}}) \mapsto I_{\text{stand}}\left(l(\theta, \underline{\widetilde{x}})\right) = \frac{l(\theta, \underline{\widetilde{x}}) - I_a}{I_b - I_a},$$

where I_a and I_b are arbitrary real values such that $I_a \leq l(\theta, \underline{\widetilde{x}}) \leq I_b$ and $I_a \neq I_b$. We have that $I_{\text{stand}}\left(l(\theta, \underline{\widetilde{x}})\right)$ is bounded and $0 \leq I_{\text{stand}}(l(\theta, \underline{\widetilde{x}})) \leq 1$.

The steps of the calculation process are written in the following manner:

1. Let $\widetilde{\theta}$ be a fuzzy parameter. We first have to calculate the log-likelihood function $l(\widetilde{\theta}; \underline{\widetilde{x}})$ shown in Eq. 19.
2. The support and the core sets defining the fuzzy number modelling the ML-estimator are composed of an infinity of values. We choose the lower and upper bounds of these sets only. The aim is to consider a reduced number of elements only. Therefore, let p, q, r and s, $p \leq q \leq r \leq s$, be the considered four elements, and $\text{supp}(\widetilde{\theta})$ and $\text{core}(\widetilde{\theta})$ be respectively the support and the core sets of $\widetilde{\theta}$. The four values p, q, r and s are then:

$$p = \min(\text{supp}(\widehat{\widetilde{\theta}})); \quad q = \min(\text{core}(\widehat{\widetilde{\theta}})); \tag{24}$$

$$r = \max(\text{core}(\widehat{\widetilde{\theta}})) \text{ and } s = \max(\text{supp}(\widehat{\widetilde{\theta}})). \tag{25}$$

In addition, the fuzzy parameter is bounded and the sets $\text{supp}(\widetilde{\widetilde{\theta}})$ and $\text{core}(\widetilde{\widetilde{\theta}})$ are not empty. This leads to conclude that the four values p, q, r and s always exist. We highlight that our intentional choice of elements is in some sense evident. Note that assuming the symmetry of the probability function, the left and right-hand sides of a log-likelihood function are monotonic and continuous.
3. Next, η has to be estimated. The bootstrap technique is suggested as described in the next section.
4. Based on the estimated parameter η, we construct the threshold values denoted by I_1, I_2, I_3 and I_4 corresponding to the chosen values p, q, r and s, respectively. The idea is to evaluate θ for each of the four values on the right-hand side of Eq. 22. The threshold values are then calculated as follows:

$$I_1 = l(p; \underline{\widetilde{x}}) - \frac{\eta}{2}; \quad I_2 = l(q; \underline{\widetilde{x}}) - \frac{\eta}{2}; \tag{26}$$

$$I_3 = l(r; \underline{\widetilde{x}}) - \frac{\eta}{2} \text{ and } I_4 = l(s; \underline{\widetilde{x}}) - \frac{\eta}{2}. \tag{27}$$

5. Next, we calculate $I_{\min}$ and $I_{\max}$, the minimum and maximum thresholds, written as:

$$I_{\min} = \min(I_1, I_2, I_3, I_4), \tag{28}$$

$$\text{and } I_{\max} = \max(I_1, I_2, I_3, I_4). \tag{29}$$

Computing $I_{\min}$ and $I_{\max}$ and including them in the calculation process are essential at this stage. The reason for that is that we want to cover the entire interval of the possible values verifying Eq. 22.
6. We find now the intersection between the log-likelihood function and the threshold values I_1, I_2, I_3 and I_4. Consider $\theta_1^{\star L}$, $\theta_2^{\star L}$, $\theta_3^{\star L}$, $\theta_4^{\star L}$ and $\theta_1^{\star R}$, $\theta_2^{\star R}$, $\theta_3^{\star R}$, $\theta_4^{\star R}$ to be the intersection abscissas. Note that the letters "L" and "R" refer to the left and right sides of a particular entity. We calculate these abscissas by solving the following equations:

$$l^L(\theta_1^{\star L}; \underline{\widetilde{x}}) = I_1 \quad \text{and} \quad l^R(\theta_1^{\star R}; \underline{\widetilde{x}}) = I_1, \tag{30}$$

$$l^L(\theta_2^{\star L};\underline{\widetilde{x}}) = I_2 \quad \text{and} \quad l^R(\theta_2^{\star R};\underline{\widetilde{x}}) = I_2, \tag{31}$$

$$l^L(\theta_3^{\star L};\underline{\widetilde{x}}) = I_3 \quad \text{and} \quad l^R(\theta_3^{\star R};\underline{\widetilde{x}}) = I_3, \tag{32}$$

$$l^L(\theta_4^{\star L};\underline{\widetilde{x}}) = I_4 \quad \text{and} \quad l^R(\theta_4^{\star R};\underline{\widetilde{x}}) = I_4. \tag{33}$$

7. Next, we compute the minimum and maximum left intersection abscissas given by

$$\theta_{\inf}^{\star L} = \inf(\theta_1^{\star L}, \theta_2^{\star L}, \theta_3^{\star L}, \theta_4^{\star L}), \tag{34}$$

$$\text{and } \theta_{\sup}^{\star L} = \sup(\theta_1^{\star L}, \theta_2^{\star L}, \theta_3^{\star L}, \theta_4^{\star L}). \tag{35}$$

The minimum and maximum right intersection abscissas are similarly written as:

$$\theta_{\inf}^{\star R} = \inf(\theta_1^{\star R}, \theta_2^{\star R}, \theta_3^{\star R}, \theta_4^{\star R}), \tag{36}$$

$$\text{and } \theta_{\sup}^{\star R} = \sup(\theta_1^{\star R}, \theta_2^{\star R}, \theta_3^{\star R}, \theta_4^{\star R}). \tag{37}$$

We remark that the left and right side intersection abscissas are single and real values.

8. The previously calculated entities are accordingly used to construct the α-cuts of the fuzzy confidence interval using the likelihood ratio method $\widetilde{\Pi}_{LR}$. We define the left and right α-cuts $(\widetilde{\Pi}_{LR})_\alpha = \left[(\widetilde{\Pi}_{LR})_\alpha^L;(\widetilde{\Pi}_{LR})_\alpha^R\right]$ in the following manner:

$$(\widetilde{\Pi}_{LR})_\alpha^L = \left\{ \theta \in \mathbb{R} \mid \theta_{\inf}^{\star L} \le \theta \le \theta_{\sup}^{\star L} \text{ and } \alpha = I_{stand}\left(l(\theta,\underline{\widetilde{x}})\right) = \frac{l(\theta,\underline{\widetilde{x}}) - I_{\min}}{I_{\max} - I_{\min}} \right\}, \tag{38}$$

$$(\widetilde{\Pi}_{LR})_\alpha^R = \left\{ \theta \in \mathbb{R} \mid \theta_{\inf}^{\star R} \le \theta \le \theta_{\sup}^{\star R} \text{ and } \alpha = I_{stand}\left(l(\theta,\underline{\widetilde{x}})\right) = \frac{l(\theta,\underline{\widetilde{x}}) - I_{\min}}{I_{\max} - I_{\min}} \right\}. \tag{39}$$

Note that Berkachy [2] gives the complete proof that the defended fuzzy confidence interval $\widetilde{\Pi}_{LR}$ verifies Definition 9. Concerning the coverage rate, it is also proven that the Eq. 23 theoretically holds.

Bootstrap Technique for the Approximation of the Likelihood Ratio and its Distribution

The bootstrap technique formally described by Efron [12] is a great tool to empirically estimate a specific sampling distribution using observed data. This technique is based on drawing a large number of samples from a primary random sample taken from an unknown distribution. This operation leads to construct a so-called bootstrap distribution of the statistic of interest. To sum up, this approach seems to estimate such distributions using random simulation-based calculation processes. The bootstrap technique has also served in fuzzy statistics. As such, Gonzalez-Rodriguez et al. [15] used it in the hypotheses testing procedure for the mean of fuzzy random variables. In the same direction, Montenegro et al. [18] concluded that a bootstrap process is considered to be computationally lighter than asymptotic ones.

In our strategy, we propose to use a bootstrap methodology to empirically estimate the distribution of the likelihood ratio LR exposed in Eq. 20, i.e. the difference of the log-likelihood function evaluated at $\underline{\widetilde{\widetilde{\theta}}}$ compared to the one evaluated at $\underline{\widetilde{\theta}}$. Berkachy and Donzé [7] has introduced two approaches to construct the bootstrap imprecise samples as follows:

1. The first one is based on simply generating with replacement D bootstrap samples. For each sample, we calculate after the needed deviance.
2. The second one is based on generating D samples by preserving the couple of location and dispersion characteristics respectively denoted by (s_0, ϵ), of the nearest symmetrical trapezoidal fuzzy numbers described in Definition 8. Note that these fuzzy numbers calculated rely on the primary data set.

Further description of both approaches remain at disposal in [7] and [2]. From [2], we can clearly see that no notable differences exist between the use of both algorithms. Although the design of both algorithms is somehow different, the obtained results seemed to be very similar. For this reason, we will detail hereafter only the second approach, considered to be more complicated than the first one, but conceptually very attractive. The algorithm based on the second bootstrap approach using the characteristics (s_0, ϵ) is then given by the following steps:

Algorithm:

1. Consider a primary sample. For each observation of this sample, calculate the set of characteristics (s_0, ϵ).
2. From the calculated set of characteristics (s_0, ϵ), randomly draw with replacement and with equal probabilities a new set of characteristics (s_0, ϵ). Construct a bootstrap sample based on this set.
3. Calculate the deviance $2\left[l(\underline{\widetilde{\theta}};\underline{\widetilde{x}}) - l(\underline{\widetilde{\theta}};\underline{\widetilde{x}})\right]^{\text{boot}}$ for each bootstrap sample.
4. Recursively repeat the Steps 2 and 3 a large number D of times. A bootstrap distribution composed of a number D of values has to be constructed.
5. Calculate η, the $(1-\delta)$-quantile of the bootstrap distribution of the statistic LR.

This algorithm mandatory requires to calculate a maximum likelihood estimator. For this task, Denoeux [10] proposed a tool based on the fuzzy EM algorithm. This latter can be computed using the `R` package `EM.Fuzzy` described in [19]. Nevertheless, this methodology presents a drawback since it produces a crisp estimator instead of a fuzzy one. Unfortunately, methods for calculating a fuzzy maximum likelihood estimator in such contexts is not yet established. For this reason, we model the obtained EM crisp-based estimator using a triangular symmetrical fuzzy number as instance. The calculated crisp element will be chosen to be the core of the modelling fuzzy number. Note that the choice of the symmetrical shape of the fuzzy number is intentional, since the purpose is to reduce as much as possible the complexity relate to this choice.

We finally highlight that all the previously described steps of the calculation procedure can be easily computed using our `R` package `FuzzySTs` described in [8]. This package is a complete user-friendly one, for which the development is made for application purposes. In addition, a detailed numerical example of the defended procedure with its interpretation can be found in [7].

Inference: Comparison of Means

Fuzzy confidence intervals are very useful in statistical inference. We propose to use these intervals in a more or less complex testing situation. Indeed, we introduce a pragmatic approach to perform a hypotheses test for comparing the means of groups using the constructed fuzzy confidence intervals. The fuzzy analysis of variance (FANOVA) is often used for this purpose as seen in Berkachy and Donzé [4], Parchami and al. [20], and Gonzalez-Rodriguez and al. [14]. We complete this analysis by proposing a test based on fuzzy confidence intervals. Our approach is as follows:

Similarly to the approach of a classical analysis of variance (ANOVA), we define the null hypothesis H_0 that the means related to the two groups are equal, against the alternative one H_1 that the pair of means is not equal, at a significance level δ. The null and alternative hypotheses H_0 and H_1 can then be written as follows:

$$H_0 : \mu_1 = \mu_2, \quad \text{against} \quad H_1 : \mu_1 \neq \mu_2,$$

where μ_1 and μ_2 are the means of the groups 1 and 2 respectively.

For the groups 1 and 2, we first construct the fuzzy confidence intervals by the likelihood ratio method denoted by $\widetilde{\Pi}_{LR_1}$ and $\widetilde{\Pi}_{LR_2}$. Our strategy of hypothesis testing is to analyse the overlapping between the fuzzy confidence intervals for each group mean. The aim is to be able to identify whether the means of groups are potentially equal or not, using these intervals. In case of perfect overlapping, we could infer that there is no difference between the means. Since the metrics described in "The $d^{\theta^\star}_{SGD}$ Metric" are seen as powerful alternative for the difference between two fuzzy sets, we propose to calculate the $d^{\theta^\star}_{SGD}$ metric of both constructed intervals, denoted by $d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_1}, \widetilde{\Pi}_{LR_2})$. The objective is then to quantify the overlapping between them. We highlight that the choice of the $d^{\theta^\star}_{SGD}$ metric is intentional since taking into consideration the shape of the fuzzy confidence intervals and its possible irregularities is crucial in this situation. In addition, a mapping into $\mathbb{R}^+$ is important on an absolute manner.

Next, we would like to define a decision rule according to the obtained overlap between both fuzzy sets. As such, we propose to "normalise" the calculated distance in order to obtain a relative ratio. Thus, by translating one set, the other remaining fixed, we calculate an optimal distance of rejection between the fuzzy confidence intervals, as the position of the intervals such that both intervals become tangent. This distance is denoted by $d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_1}, \widetilde{\Pi}_{LR_2})_{\text{opt}}$. The following statistic R given by:

$$R = \frac{d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_1}, \widetilde{\Pi}_{LR_2})}{d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_1}, \widetilde{\Pi}_{LR_2})_{\text{opt}}}, \tag{40}$$

will help us to reject or not the null hypothesis. The decision rule can then be written as follows:

Decision rule: The statistic R belongs to the interval [0; 1]. The rules are then:

- The closer the statistic R is to the value 0, the strongest we do not reject the null hypothesis H_0.
- The closer the statistic R is to the value 1, the strongest we reject the null hypothesis H_0.

Simulation Study

In [7], we have showed a simulation study illustrating the use of the two defended bootstrap algorithms in the process of calculation of the fuzzy confidence intervals. This study is based of randomly generating data sets taken from a normal distribution $N(5, 1)$ and composed by $N = 50$, 100 and 500 observations. The observations are then modelled by triangular symmetrical fuzzy numbers of spread 2.

Following the well-described procedure, the fuzzy confidence intervals by the likelihood ratio for the theoretical of the constructed data sets were computed at the confidence level $1 - \delta = 1 - 0.05$. The algorithm presented in "Bootstrap technique for the approximation of the likelihood ratio and its distribution" has been used to estimate the bootstrapped quantile η.

In addition, since the number of iterations did not really influence the outcome of the calculations, $D = 1000$ iterations were considered for all our calculations. Concerning

Table 1 The 95%-quantiles of the bootstrapped distribution of LR and the corresponding fuzzy confidence intervals by the likelihood ratio (data set of 500 observations)—case of a data set taken from a normal distribution $N(5, 1)$ modelled using triangular symmetrical fuzzy numbers at 1000 iterations

Bootstrap quantiles				
Sample size		$N = 50$	$N = 100$	$N = 500$
Bootstrap quantile using the sample mean		1.802	1.845	2.118
Bootstrap quantile using the ML-estimator (spread 2)		1.854	1.971	2.201
Bootstrap quantile using the ML-estimator (spread 1)		1.563	1.671	1.864
Fuzzy confidence intervals				
	Support set		Core set	
	Lower	Upper	Lower	Upper
Fci using the sample mean	3.991	5.996	4.945	5.042
Fci using the ML estimator (spread 2)	3.803	6.184	4.795	5.193
Fci using the ML estimator (spread 1)	4.303	5.685	4.797	5.191

the crisp-based estimators calculated using the fuzzy EM algorithm, we have considered the following two fuzzy numbers to model the estimators:

- the first one is a triangular symmetrical fuzzy number of spread 2;
- the second one is a triangular symmetrical fuzzy number of spread 1.

We would like to explore the influence of the degree of fuzziness in the modelling procedure of the estimators on the constructed fuzzy confidence intervals. For sake of comparison, we additionally use the fuzzy sample mean as a fuzzy estimator.

We show in Table 1 the 95%-quantiles of the bootstrapped distribution of the likelihood ratio, where data sets of size 50, 100 and 500 are considered. It is clear to see that the quantiles corresponding to the considered sample sizes are in some sense very close. We could directly remark also that modelling the ML-estimator using less fuzziness (fuzzy number with spread 1) leads to a lower quantile, compared to modelling the ML-estimator using greater fuzziness (fuzzy number with spread 2).

Based on the boostrapped quantiles shown in Table 1, we now calculate the fuzzy confidence intervals using the likelihood ratio method following the instructions given in "Procedure". For sake of simplicity, we will develop the case with $N = 500$ observations only for the construction of confidence intervals. Table 1 gives the lower and upper bounds of the support and the core sets of the calculated fuzzy confidence intervals.

Concerning the interpretation of the choice of degree of fuzziness related to the ML-estimators, it is clear to see that less fuzziness leads a smaller support set of the calculated confidence interval. In other terms, this choice affects the obtained fuzzy confidence interval. Therefore, carefully modelling the ML-estimator is crucial.

By traditional fuzzy tools, a fuzzy confidence interval defined in the same settings is given by $\widetilde{\Pi} = (3.907, 4.907, 5.080, 6.080)$. An important conclusion of the difference between the traditional and the defended fuzzy confidence intervals is that the core sets are slightly larger in the case of bootstrap intervals using the ML-estimators. Note that for these intervals, interpreting the spread of the support sets is in some sense difficult since they are affected by the degree of fuzziness of the ML-estimator. In case the fuzzy sample mean is used as an estimator, the obtained fuzzy confidence interval has tighter support and core sets compared to the traditional fuzzy confidence interval.

Simulation Study on Coverage Rates

In [7], we have conducted a simulation study on coverage rates corresponding to the fuzzy confidence intervals calculated using the likelihood ratio method. A large number of data sets composed of $N = 100$, 500 and 1000 observations are generated. We consider these data sets to be uncertain and we model each observation by a triangular symmetrical fuzzy number with a spread 2. The objective of this study is to estimate for the mean, fuzzy confidence intervals by the likelihood ratio method for one side, and the traditional fuzzy ones from another one, at the confidence level $1 - \delta = 1 - 0.05$, and consequently calculate the coverage rates of these intervals in order to compare them. We note that the ML-estimators were modelled by fuzzy numbers of spreads 1 and 2. Similarly to the previous study, the fuzzy sample mean was also used for the sake of comparison only.

Some important conclusions of the comparison of coverage rates mentioned in this study are as follows:

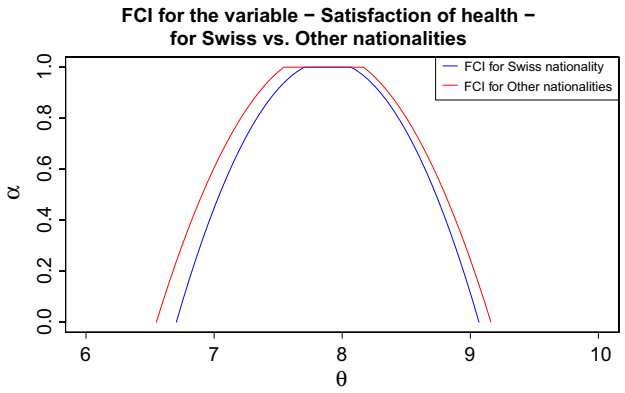
- The difference between the coverage rates of the bootstrap fuzzy confidence intervals and the ones of the traditional fuzzy interval is very slight.
- The coverage rate of the fuzzy confidence intervals by the likelihood method where the fuzzy sample mean is used as an estimator, is the same as the rate for the traditional fuzzy one.
- All the coverage rates of LR fuzzy confidence guarantee the required theoretical 95% confidence level.
- The fuzziness of the ML-estimators do not influence the coverage rates of the calculated fuzzy confidence intervals.

Application on SILC 2017

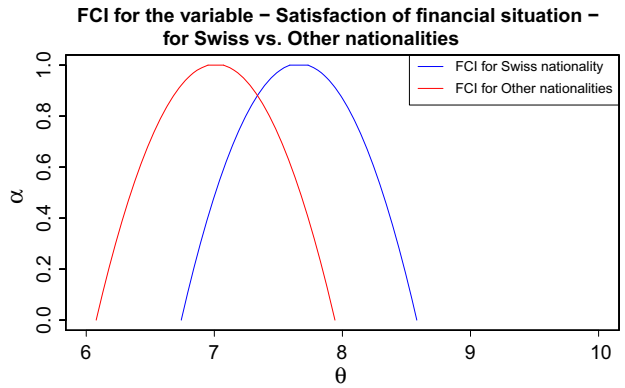
The Swiss SILC surveys are large and complex surveys conducted every year by the Swiss Federal Statistical Office [21] on Statistics on Income and Living Conditions in Switzerland. In our application, we use the data from the 2017 edition. After selecting only the active population, i.e. persons which age greater or equal to 18, and persons who are respondent for the whole household, we get a sample of 8120 observations. In this study, we do not apply any weighting scheme. We consider in particular two variables describing respectively the health situation (`PW5020`) and the financial condition (`HQ5010`). Both variables are coded on a Likert scale from 0 to 10, where the value 0 means not satisfied. We are mainly interested in the difference between the group of Swiss nationality from one side, and the group of other nationalities from another one. As such, we perform the defended calculation approaches for the variables `PW5020` and `HQ5010` by each of both groups.

First of all, we would like to construct the fuzzy confidence intervals by the likelihood ratio method at the confidence level 95% for the variables `PW5020` and `HQ5010` as previously discussed. Each modality of these variables is modelled by a triangular symmetrical fuzzy number of spread 2. In other words, the value 3 is for example modelled by the triangular fuzzy number (2, 3, 4). It is the same for all the other modalities of both variables. The obtained intervals for both variables are shown in Fig. 1. The support and the core sets of these intervals are also given in Table 2. We highlight that the bootstrap algorithm presented in "Bootstrap technique for the approximation of the likelihood ratio and its distribution" is used.

We focus the analyses of this section on two main axes described as follows:



(a) `Health situation - PW5020`



(b) `Satisfaction of financial situation - HQ5010`

Fig. 1 Fuzzy confidence intervals (FCI) of the variables `Health situation—PW5020` (**a**) and `Satisfaction of financial situation—HQ5010`(**b**) at the confidence level 95%

1. The influence of the variation in the fuzziness of the ML-estimator from one side, and the influence of the variation of the confidence level from another one on the constructed FCI. These analyses are performed based on the variable `PW5020`;
2. The hypothesis test on the means of the groups Swiss vs. Other nationalities using the constructed fuzzy confidence intervals as proposed in "Inference: comparison of means". This analysis is based on the variable `PW5020` and on the variable `HQ5010`.

Fuzziness vs. Randomness

As previously mentioned, the objective of a first set of analyses is to investigate the influence of the variation of fuzziness related to the ML-estimator from one side, and the variation of the confidence level from another one.

In our context, the ML-estimator is calculated by the EM-algorithm defined in the fuzzy environment as seen in [10]. This approach unfortunately leads to a crisp-based estimator, and we consequently need to re-fuzzify it. For this reason,

Table 2 The fuzzy confidence intervals by the likelihood ratio for the groups "Swiss" and "Other" nationalities of the variables `Health situation—PW5020` and `Satisfaction of financial situation—HQ5010` at the confidence level 95% using triangular symmetrical fuzzy numbers of spread 2

`Health situation—PW5020`				
Nationality	Support set		Core set	
	Lower	Upper	Lower	Upper
Swiss	6.705	9.068	7.703	8.069
Other	6.549	9.161	7.544	8.166
`Financial situation—HQ5010`				
Nationality	Support set		Core set	
	Lower	Upper	Lower	Upper
Swiss	6.742	8.581	7.589	7.732
Other	6.078	7.942	6.951	7.072

Table 3 The fuzzy confidence intervals by the likelihood ratio for the groups "Swiss" and "Other" nationalities of the variable `Health situation—PW5020` at the confidence level 95% using triangular symmetrical fuzzy numbers of spread 2

Swiss nationality				
	Support set		Core set	
	Lower	Upper	Lower	Upper
Fci using the ML estimator (spread 2)	6.705	9.068	7.703	8.069
Fci using the ML estimator (spread 1)	7.205	8.568	7.703	8.069
Other nationality				
	Support set		Core set	
	Lower	Upper	Lower	Upper
Fci using the ML estimator (spread 2)	6.549	9.161	7.544	8.166
Fci using the ML estimator (spread 1)	7.048	8.662	7.649	8.161

we have proposed to model this estimator by a triangular symmetrical fuzzy number of spread 2. Such construction is in some sense natural. We are interested in investigating the influence of such choice on the constructed confidence intervals. For this reason, we propose on a second stage to model these estimators by triangular symmetrical fuzzy numbers of spread 1 in order to understand the influence of such variation.

In Tables 3 and 4, we show the support and the core sets of the obtained confidence intervals by the described procedure. By these tables, we could clearly confirm the conclusion of the simulation study performed in "Simulation study". In other words, independently from the chosen groups of the variable `PW5020`, the degree of fuzziness of the fuzzy confidence interval is strongly affected by the choice of fuzziness of the ML-estimator. For the same confidence level, more fuzziness of the ML-estimator leads to a greater support set of the constructed confidence interval. Therefore, it would be ideal if a complete fuzzy-based approach of calculation of ML-estimator exists.

In terms of confidence levels, Table 3 shows the fuzzy confidence intervals at the confidence level 95% while Table 4 gives the ones at the confidence level 70%. By comparing both tables, one could clearly remark that no important variation in terms of spread of the constructed fuzzy intervals, is depicted. A very small fluctuation is seen only. A general conclusion that we could propose is that the randomness in the data is less influencing the constructed fuzzy confidence intervals than the fuzziness in the data and in the ML-estimator.

Test on Equality of Means

We are now interested in comparing the means of groups based on the interpretation of fuzzy confidence intervals. We would like then to determine whether the means of the two

Table 4 The fuzzy confidence intervals by the likelihood ratio for the groups "Swiss" and "Other" nationalities of the variable `Health situation—PW5020` at the confidence level 70% using triangular symmetrical fuzzy numbers of spread 2

Swiss nationality				
	Support set		Core set	
	Lower	Upper	Lower	Upper
Fci using the ML estimator (spread 2)	6.715	9.058	7.713	8.059
Fci using the ML estimator (spread 1)	7.214	8.559	7.713	8.060
Other nationality				
	Support set		Core set	
	Lower	Upper	Lower	Upper
Fci using the ML estimator (spread 2)	6.570	9.139	7.565	8.145
Fci using the ML estimator (spread 1)	7.069	8.641	7.629	8.081

Table 5 The results of the classical analysis of variance (ANOVA) and the fuzzy analysis of variance (FANOVA) for the variable `Health situation—PW5020`

ANOVA				
	Df	Sum Sq	*F* value	*P* value
Factor	1	1	0.267	0.605
Residuals	7423	24437		
FANOVA				
	Df	Sum Sq	*F* value	*P* value
Factor	1	0.854	0.279	0.597
Residuals	7423	22740.454		

Table 6 The results of the classical analysis of variance (ANOVA) and the fuzzy analysis of variance (FANOVA) for the variable `Satisfaction of financial situation— HQ5010`

ANOVA				
	Df	Sum Sq	*F* value	*P* value
Factor	1	484	111.7	0^{***}
Residuals	8084	35052		
FANOVA				
	Df	Sum Sq	*F* value	*P* value
Factor	1	451.175	112.049	0^{***}
Residuals	8084	32551.067		

groups "Swiss" versus "Other" nationality of the variables `PW5020` and `HQ5010` are equal to each other, or not. Such hypotheses test are usually made by the well-known analysis of variance (ANOVA) in the classical statistical theory, or by the fuzzy analysis of variance (FANOVA) in the fuzzy set theory. For this reason, we perform the ANOVA as well as the FANOVA for the variables `PW5020` and `HQ5010` by nationality. The summaries are given in Tables 5 and 6 respectively.

Table 5 shows that for the variable `PW5020`, the null hypothesis of equality of means is not rejected with a p-value of 0.605 by ANOVA and 0.597 by FANOVA. Let us now

perform the same hypothesis test but using the fuzzy confidence intervals. First of all, the Fig. 1a is a great tool to visualise the potential equality of means. From this figure, we could clearly see that the support and the core sets of both intervals overlap completely. In other terms, we could have an evident suspicion of the equality between the perception of health situation by Swiss natives or Other nationalities. We would like to apply now the testing procedure described in "Inference: Comparison of means". Thus, we calculate the actual distance between the fuzzy confidence interval by the likelihood ratio for the group "Swiss", i.e. $\widetilde{\Pi}_{LR_{CH}}$, and the one for the group "Other", i.e. $\widetilde{\Pi}_{LR_{OT}}$, using the metric $d^{\theta^\star}_{SGD}$. This distance is given by

$$d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_{CH}}, \widetilde{\Pi}_{LR_{OT}}) = 0.111,$$

for which the optimal distance between these two intervals can be found by the translation technique such that

$$d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_{CH}}, \widetilde{\Pi}_{LR_{OT}})_{\text{opt}} = 2.489.$$

The statistic R can then be computed using the Eq. 40. We get:

$$R = 0.045.$$

Decision rule: Since the statistic R is very close to the value 0, we strongly not reject the null hypothesis of equality of means between both groups. This conclusion confirms our ANOVA and FANOVA results.

We perform the same analysis for the variable `HQ5010`. By Table 6, it is evident to see that the null hypothesis of equality of means is strongly rejected in the ANOVA and the FANOVA with a P value of 0. If we consider the fuzzy confidence intervals by the likelihood ratio for both studied groups shown in Fig. 1b, we remark that both intervals partially overlap. In this case, the overlapping is exclusively a matter of the support sets of the intervals. The core sets do not overlap. By the hypotheses testing procedure presented in "Inference: Comparison of means", we calculate the distance $d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_{CH}}, \widetilde{\Pi}_{LR_{OT}})$ and we get:

$$d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_{CH}}, \widetilde{\Pi}_{LR_{OT}}) = 0.652.$$

In this case, the maximal distance between the fuzzy confidence interval by the likelihood ratio for the group "Swiss", i.e. $\widetilde{\Pi}_{LR_{CH}}$, and the one for the group "Other", i.e. $\widetilde{\Pi}_{LR_{OT}}$ is

$$d^{\theta^\star}_{SGD}(\widetilde{\Pi}_{LR_{CH}}, \widetilde{\Pi}_{LR_{OT}})_{\text{opt}} = 1.852.$$

Therefore, the statistic R is calculated as:

$$R = 0.352.$$

Decision rule: Since the statistic R is relatively far from the value 0, we tend to strongly reject the null hypothesis of equality of means between both groups, confirming our ANOVA and FANOVA results.

To sum up, a clear conclusion of the use of such decision rule is that the closer the statistic R to the value 0, the strongest we do not reject the hypothesis of equality of means. However, once we move away from the neighbourhood of 0, we strongly enter into the rejection region. By our decision rule, one could reject or not the hypothesis of equality of means with confidence. Accordingly, a final remark is that the use of the statistic R has to be prudent since R is in some sense an indicator and not a threshold of rejection. The adopted criteria in our methodology is mainly related to the shape and the position of the fuzzy confidence interval.

Conclusion

This study proposed a complete practical procedure of estimation of fuzzy confidence intervals using the likelihood ratio method. The bootstrap technique was also used. A complex bootstrap algorithm based on preserving the location and dispersion measures related to the $d^{\theta^\star}_{SGD}$ metric is exposed. Such calculations are often seen as expensive in terms of computational burden. Our procedure is well-designed in order to avoid such complexities. Based on the developed fuzzy confidence intervals, we introduced a hypothesis test for the equality of means of two groups with its corresponding decision rule.

Our approaches have been applied on the Swiss SILC surveys where two main axes of analyses were developed: the influence of the fuzziness versus the randomness of the data and of the maximum likelihood estimator on the confidence intervals from one side, and the application of the hypotheses testing procedure for comparing the mean of the group "Swiss nationality" to that of the group "Other nationalities" for the variables Satisfaction of health situation and Satisfaction of financial situation.

Main conclusions of these analyses are that the randomness in the data is less influencing the fuzzy confidence intervals than the fuzziness in the data and of the ML-estimator. Furthermore, one could clearly see that fuzzy confidence intervals can be considered as a proper tool to perform a preliminary analysis for the comparison of means. Thus, the introduced statistic is in some sense an indicator of rejection of the hypothesis of equality of means of two groups.

Furthermore, we have to mention that it is often difficult to conduct a hypothesis test based on fuzzy confidence intervals. It needs deeper knowledge of the hypotheses, the considered estimators and distributions as instance. Based on our constructed procedure, we managed to propose a conceptually different way of comparison between two means. This method shows also an indicator of overlapping, which can be strongly helpful in the decision making process. From

another side, the defended hypothesis test by the likelihood ratio method is an interesting alternative to the classical test. Our approach is practicable, direct, and easily reproducible using our R toolbox given in Berkachy and Donzé [8]. This latter is well conceived and implemented in a way of having a good performance in terms of time and computational burden needed for numerical purposes. Multiple benchmark tests have been previously performed to ensure its performance. Thus, this R package is a great tool, since it can be easily used to carry out such calculations of fuzzy confidence intervals in a user-friendly environment. We highlight that fuzzy hypotheses can also be taken into consideration in the calculation of these fuzzy intervals by our toolbox.

Finally, the encountered problem is mainly in the fuzzification of the crisp-based maximum likelihood estimator. An approach leading to a fuzzy-based one is very welcome. From another side, refining our decision rule regarding testing the hypotheses of equality of means could also be interesting to investigate in future research.

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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